DISCRETE SCALE INVARIANCE IN TURBULENCE?

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Based on theoretical argument and experimental evidence, we conjecture that structure functions of turbulent times series exhibit log-periodic modulations decorating their power law dependence. In order to provide ironclad experimental evidence, we stress the need for novel methods of averaging and propose a novel "canonical" averaging scheme for the analysis of structure factors of turbulent flows. The strategy is to determine the scale r_c at which the dissipation rate is the largest in a given turn-over time series. This specific scale r_c translates into a specific "phase" in the logarithm of the scale which, when used as the origin, allows one to phase up the different measurements of a structure factor $S_p(r) = A_p(\bar{\epsilon} r)^{p/3}$ in different turn-over time realizations. We expect, as in Laplacian growth and in rupture, that the log-periodic oscillations will be reinforced by this canonical averaging. Demonstrating unambiguously the presence of log-periodicity and thus of discrete scale invariance (DSI) in turbulent time-series would provide an important step towards a direct demonstration of the Kolmogorov cascade or at least of its hierarchical imprint.

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Scaling laws provide a quantification of the complexity of turbulent flows. For instance, the second order longitudinal structure function has the form $\langle (v_r)^2 \rangle = C_K (\bar{\epsilon} r)^{2/3}$, according to Kolmogorov 1941 [1] where r, which lies in the inertial range, is the scale at which velocity differences are measured, and $\bar{\epsilon}$ is the mean rate of energy dissipation per unit mass. Dimensional analysis shows that $\langle (v_r)^2 \rangle = (\bar{\epsilon} r)^{2/3} F(\Re, r/L)$, where $F(x, y)$ is a universal function to be determined, L is the external or integral scale. Kolmogorov's assumption is that, for the Reynolds number $\Re \to \infty$ and $r/L \to 0$, $F(x, y)$ goes to a constant C_K. This is the so-called complete similarity of the first kind [2] with respect to the variables $\Re$ and r/L.

The existence of the limit of $F(\Re, r/L \to 0)$ has first been questioned by L.D. Landau and A.M. Obukhov, on the basis of the existence of intermittency – fluctuations of the energy dissipation rate about its mean value $\bar{\epsilon}$. Indeed, Barenblatt's classification leads to the possibility of an *incomplete similarity* in the variable r/L. This would require the absence of a finite limit for $F(\Re, r/L)$ as $r/L \to 0$, and leads in the simplest case to the form $\langle (v_r)^2 \rangle = C_K (\bar{\epsilon} r)^{2/3} \left(\frac{r}{L}\right)^{\alpha}$, where α is the so-called intermittency exponent, believed to be small and positive. If α is real, this corresponds to a similarity of the second kind [2]. Incomplete self-similarity [3,4] may stem from a possible $\Re$-dependence of the exponents. The case where α is complex, leading to $\langle (v_r)^2 \rangle = C_K (\bar{\epsilon} r)^{2/3} \left(\frac{r}{L}\right)^{\alpha_R} \cos[\alpha_I \log(r/L)]$, could be termed a similarity of the third kind, characterized by the absence of limit for $F(\Re, r/L)$ and accelerated (log-periodic) oscillations. To our knowledge, Novikov has been the first to point out in 1966 that structure functions in turbulence should contain log-periodic oscillations [5]. His argument was that if an unstable eddy in a turbulent flow typically breaks up into two or three smaller eddies, but not into 10 or 20 eddies, then one can suspect the existence of a prefered scale factor, hence the log-periodic oscillations. They have been repeatedly observed but do not seem to be stable and depend on the nature of the global geometry of the flow and recirculation [1,6] as well as the analyzing procedure.

The theory of complex exponents and log-periodicity has advanced significantly [7] in the last few years. Complex exponents reflect a discrete scale invariance (DSI), i.e. the fact that dilational symmetry occurs only under magnification under special factors, which are arbitrary powers λ^n of a prefered scaling ratio λ. Complex exponents have been studied in the eighties in relation to various problems of physics embedded in hierarchical systems. In the context of turbulence, shell models construct explicitely a discrete scale invariant set of equations whose solutions are marred by unwanted log-periodicities. Only recently has it been realized that discrete scale invariance and its associated complex exponents may appear "spontaneously" in euclidean systems, i.e. without the need for a pre-existing hierarchy. Systems that have been found to exhibit self-organized DSI are Laplacian growth models [8], rupture in heterogeneous systems [9], earthquakes [10], animals [11] (a gen-

eralization of percolation) among many other systems. In addition, general field theoretical arguments [11] indicate that complex exponents are to be expected generically for out-of-equilibrium and/or quenched disordered systems. This together with Novikov's argument suggest to revisit log-periodicity in turbulent signals. Demonstrating unambiguously the presence of log-periodicity and thus of DSI in turbulent time-series would provide an important step towards a direct demonstration of the Kolmogorov cascade or at least of its hierarchical imprint. For partial indications of log-periodicity in turbulent data, we refer the reader to fig. 5.1 p.58 and fig. 8.6 p.128 of Ref. [1], fig.3.16 p. 76 of Ref. [12], fig.1b of Ref. [13] and fig. 2b of Ref. [14].

It is a common observation that the oscillations, if any, "move" when changing the length of the signal over which the averaging is carried out. They thus have the aspect of noise. However, previous numerical simulations on Laplacian growth models [8,15] and renormalization group calculations [11] have taught us that the presence of noise modifies the phase in the log-periodic oscillations in a sample specific way leading to a "destructive interference" upon averaging. In the turbulence context, we propose that one realization corresponds approximately to a signal measured over one turn-over time scale L/v_L. In contrast to this sample specific phase dependence, we stress that the prefered scaling ratio λ has universal properties.

It is thus important to carry out an analysis on each sample realization separately, without averaging, as has been demonstrated to work for other systems [8]. An enticing alternative is to introduce a new averaging scheme that does not destroy the oscillations. The standard averaging procedure, that we could term "Grand canonical" [16], is known to introduce spurious sample-to-sample fluctuations of relative amplitude proportional to $L^{-d/2}$ in d dimensions. In contrast, the concept of "canonical" averaging [16] consists in identifying, for each realization, the corresponding specific value of the critical control parameter K_c^R. The natural control parameter then becomes $\Delta = (K - K_c^R)/K_c^R$ and the averaging can then be performed over the different samples with the same Δ. This should then lead in principle to a "rephasing" of the log-oscillations. This "canonical averaging" has been demonstrated for log-periodic signatures of the acoustic emission precursors prior to rupture and in Laplacian growth models [15].

We propose to adapt this "canonical" averaging scheme to the analysis of structure functions of turbulent flows. There are probably several possible schemes to implement it. Let us suggest here one based on the energy dissipation rate. The strategy is to look for a reference quantity that is specific to a given turn-over time realization. For critical phenomena, a natural candidate is the susceptibility whose maximum determines the sample specific critical point location K_c^R [16,15]. For turbulence, we suggest to determine the scale r_c at which the dissipation rate is the largest in a given turn-over time series. This can be derived by a direct measurement of the velocity gradient at small scales or from the third-order structure function $S_3(r) = -\frac{4}{5}\bar{\epsilon}r$, which obeys the exact four-fifth Kolmogorov law under a set of assumptions [1]. This specific scale r_c translates into a specific "phase" in the logarithm of the scale which, when used as the origin, allows one to phase up the different measurements of a structure function $S_p(r) = A_p(\bar{\epsilon}r)^{p/3}$ in different turn-over time realizations. We expect, as in Laplacian growth models and in rupture, that the log-periodic oscillations will be reinforced by this canonical averaging.

What could be the mechanism that creates these characteristic scales? There are undoubtly the integral length L and the scales associated to the dissipation range. But what could produce an approximate geometrical series of scales in the inertial range? We conjecture two possible routes. The first one is inspired from a recent discovery that the continuous nonlinear Einstein partial differential equations of general relativity in the presence of a scalar field self-interacting through gravitation may generate a log-periodic spectrum of black hole masses with develop according to a log-periodic self-similar time dynamics [17]. The mechanism might result from the existence of a limit cycle in the renormalization group description of a field close to the negative density limit (in turbulence, could this be obtained from a negative effective viscosity?). The other route is that scale invariant equations that present an instability at finite wavevector k decreasing with the field amplitude may generate naturally a spectrum of internal scales. An example is $\frac{\partial v}{\partial t} = -2v\frac{\partial^2 v}{\partial x^2} - v^2\frac{\partial^4 v}{\partial x^4}$. This equation is scale invariant in the sense that if $v(t,x)$ is a solution, then $\gamma^2 v(t, \gamma x)$ is also a solution for arbitrary γ. A linear stability analysis shows that a mode $v_0 e^{\sigma t} e^{ikx}$ grows with $\sigma = 2v_0k^2 - v_0^2k^4$, i.e. the most unstable mode occurs at finite $k_{m.u.} = \frac{1}{\sqrt{v_0}}$. Thus, a finite characteristic scale appears that is completely controlled by the amplitude of the field. Starting from an approximate homogeneous level v_0, the instability produces a large scale $\frac{2\pi}{k_{m.u.}} = 2\pi\sqrt{v_0}$. As dips in the field develop, the amplitude there decreases and the corresponding instabilities will create smaller length scales, and so on. Preliminary simulations [18] confirm this intuitive picture: the resulting DSI field is seen to result from a cascade of instabilities with characteristic wavelengths controlled by the amplitude.

I hope that these conjectural ideas will stimulate further works on these fascinating log-periodic structures in turbulent signals. I am grateful to U. Frisch, L. Gil, N. Goldenfeld, A. Johansen, A. Noullez and G. Simms for discussions.

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